

450A Political Methodology I

Lecture 5: Regression with Two Regressors

4. Interaction Terms and Polynomials

Stanford University

Interaction Terms

Interaction between Dummy and Continuous Variables

Interaction between Two Continuous Variables

Interaction Terms: Common Pitfalls and Solutions

Polynomials

Why Interaction Terms?

- Interaction terms will allow you to let the **slope** on one variable vary as a function of another variable
- Interaction terms are central in regression analysis to:
 - Model and test conditional hypotheses (e.g. do the returns to education vary by gender?)
 - Make the model of the conditional expectation function more realistic by letting coefficients vary across subgroups
- We can interact:
 - two or more dummy variables
 - dummy variables and continuous variables
 - two or more continuous variables
- Interactions often confuse researchers and mistakes in use and interpretation occur frequently (even in top journals).

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Interaction between Dummy and Continuous Variables

- We return to the example of Lectures 5-1 and 5-2. How does **executive corruption** (Y , 0–100) vary with **court independence** and with whether a country is a **democracy**? We use 179 countries in 2024.
- Consider a continuous and a dummy variable:

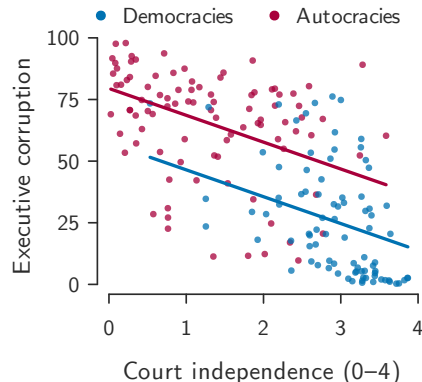
$X_{court,i}$ = high court independence (0 = never independent of the government, 4 = always)

$$Z_{dem,i} = \begin{cases} 1 & \text{if country } i \text{ is a democracy (V-Dem electoral democracy index above 0.5)} \\ 0 & \text{if country } i \text{ is an autocracy} \end{cases}$$

- Let's examine the interplay between these variables using regression.

Corruption, Court Independence, and Democracy

- Countries with more independent courts have less executive corruption.
- Democracies have less executive corruption on average (26 vs. 65).
- Democracies also tend to have more independent courts (2.9 vs. 1.3 on average).
- We fit: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_{court} + \hat{\beta}_2 Z_{dem}$
- It isn't obvious that the relationship between corruption and court independence is the same for both groups.



Source: V-Dem v16, 2024 (executive corruption index $\times$ 100; high court independence).

Corruption, Court Independence, and Democracy

We may want to allow the slopes to differ by including an interaction term and thereby generalize our regression equation:

$$Y = \beta_0 + \beta_1 X_{court} + \beta_2 Z_{dem} + \beta_3 (X_{court} \cdot Z_{dem}) + u$$

What does this mean? When $Z_i = 0$ (autocracies):

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{court} + \hat{\beta}_2 Z_{dem} + \hat{\beta}_3 (X_{court} \cdot Z_{dem})$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{court}$$

When $Z_i = 1$ (democracies):

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{court} + \hat{\beta}_2 Z_{dem} + \hat{\beta}_3 (X_{court} \cdot Z_{dem})$$

$$\hat{Y}_i = (\hat{\beta}_0 + \hat{\beta}_2) + (\hat{\beta}_1 + \hat{\beta}_3) X_{court}$$

We fit two lines with different slopes AND different intercepts.

Corruption, Court Independence, and Democracy

- From R, we obtain estimates $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$:

Cons	Court	Dem	Court:Dem
75.3	-7.7	2.7	-10.3

- Autocracies:

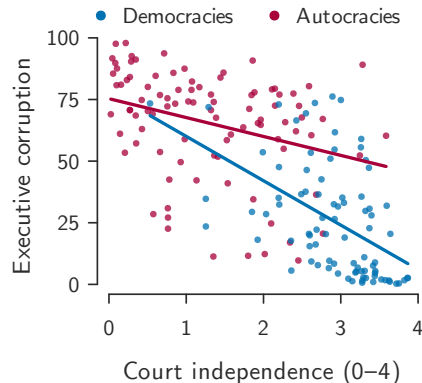
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X_{court}$$

$$\hat{y} = 75.3 - 7.7 X_{court}$$

- Democracies:

$$\hat{y} = (\hat{\beta}_0 + \hat{\beta}_2) + (\hat{\beta}_1 + \hat{\beta}_3) X_{court}$$

$$\hat{y} = 78.0 - 18.0 X_{court}$$



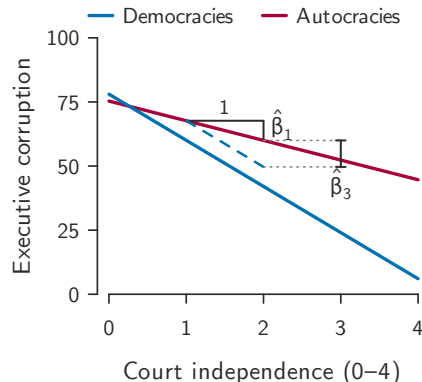
Corruption, Court Independence, and Democracy

Our prediction equation is:

$$\hat{Y}_i = 75.3 - 7.7 X_{court,i} + 2.7 Z_{dem,i} - 10.3(X_{court,i} \cdot Z_{dem,i})$$

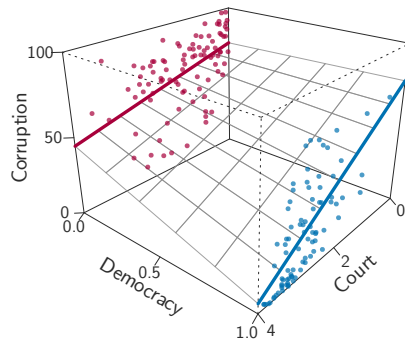
Where do these quantities appear?

- $\hat{\beta}_0 = 75.3$ is the intercept for the prediction line for autocracies.
- $\hat{\beta}_1 = -7.7$ is the slope for autocracies.
- $\hat{\beta}_2 = 2.7$ is the vertical distance between intercepts, at $X_{court} = 0$, where there are no democracies (the least independent court in a democracy scores 0.53).
- $\hat{\beta}_3 = -10.3$ is the difference in slopes.



Corruption, Court Independence, and Democracy

- Again, these observations are located in three-dimensional space, which we can view from different angles.
- When we include an interaction term, the regression surface is no longer a simple plane; it is twisted in a certain way.
- If we slice through the surface holding one of the variables fixed, the slice is still linear.



Do $\hat{\beta}_1$ and $\hat{\beta}_2$ still represent marginal effects?

Worksheet Q1

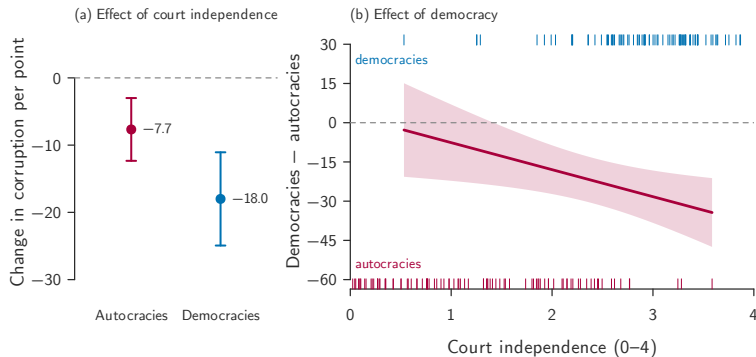
Let X_{court} be high court independence (0–4) and $Z_{dem} \in \{0, 1\}$ indicate democracy. Estimate

$$Y = \beta_0 + \beta_1 X_{court} + \beta_2 Z_{dem} + \beta_3 (X_{court} \cdot Z_{dem}) + u.$$

- 1 Derive (a) the conditional marginal effect of court independence at regime type $Z = z$ and (b) the conditional marginal effect of democracy at court independence $X = x$.
- 2 Using the delta method, give $\text{Var}(\widehat{\text{ME}}_{court}(z))$ and $\text{Var}(\widehat{\text{ME}}_{dem}(x))$, where $\widehat{\text{ME}}_{court}(z) = \hat{\beta}_1 + \hat{\beta}_3 z$ and $\widehat{\text{ME}}_{dem}(x) = \hat{\beta}_2 + \hat{\beta}_3 x$.
- 3 Explain why omitting the covariance term biases the standard errors.

Corruption, Court Independence, and Democracy

To communicate interactions effectively it is often helpful to draw **conditional marginal effect plots** that visualize the marginal effect of one variable at different levels of the other variable.



(a) Independent courts go with less executive corruption in both regime types, and more so in democracies.

(b) Drawn only where both groups have data (0.53 to 3.59). The line would cross zero at 0.26, where there are no democracies.

Back to Lecture 5-3: Leave, Scotland, and No Qualifications

In Lecture 5-3, Leave rose with NoQual (% of residents aged 16+ with no qualifications, 2011) more steeply in England & Wales than in Scotland. One interaction model gives both lines:

```
> summary(lm(Leave ~ Scotland * NoQual, data = D))
```

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	21.32652	1.53389	13.904	< 2e-16	***
Scotland	1.66270	7.22182	0.230	0.81804	
NoQual	1.47473	0.06667	22.119	< 2e-16	***
Scotland:NoQual	-0.87536	0.26714	-3.277	0.00115	**

- Slope in England & Wales: 1.47473. Slope in Scotland: $1.47473 - 0.87536 = 0.59937$. These are the two separate lines of Lecture 5-3 (1.47 and 0.60).
- The interaction tests whether the slopes differ: $t = -3.28$, $p = 0.001$.
- The Scotland coefficient (1.66) is the gap at $\text{NoQual} = 0$, far below every area (the lowest is 6.7%).

Source: Electoral Commission (2016); 2011 Census (ONS, National Records of Scotland). Data as in Lecture 5-3.

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Polynomials

We can generalize to interactions with two continuous variables:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \cdot X_2) + u$$

What does this imply?

- The marginal “effect” of X_1 is different for each value of X_2 .
- The marginal “effect” of X_2 is different for each value of X_1 .

Example: Interacting continuous variables

Let's illustrate it with the Brexit vote of Lecture 5-3, following Becker, Fetzer & Novy (2017), "Who Voted for Brexit?" *Economic Policy* 32(92): 601–650.

Outcome is the **Leave share** in the 2016 referendum, in the 380 counting areas of Great Britain.

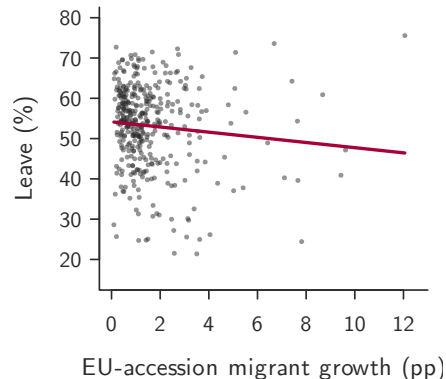
They ask whether the arrival of migrants from Eastern Europe went with a larger Leave vote where more residents had no qualifications.

The variables are defined as follows:

- `leave` is Leave as a percentage of valid votes.
- `noqual` is the percentage of residents aged 16–74 with no qualifications (or only "other" qualifications of unknown level) in 2001, as in Becker et al. (Lecture 5-3 used the 2011 share of residents aged 16+.)
- `growth` is the growth in residents born in the 12 countries that joined the EU in 2004 and 2007, from 2001 to 2011, as percentage points of the 2001 population. Zero means no net change in that group.

Leave, Qualifications, and Migration

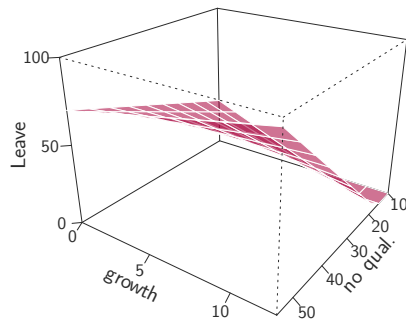
- The bivariate relationship between Leave and no qualifications is positive (1.09 points per percentage point).
- The bivariate relationship between Leave and migrant growth is weak (-0.64 , $t = -1.9$). Growth is skewed. The median area grew by 1.1 points, Boston by 12.1.
- Becker et al. ask whether there is an interaction between the two variables.



Source: Electoral Commission (2016); Census 2001 and 2011 (ONS, National Records of Scotland). Design of Becker, Fetzer & Novy (2017), Table 6.

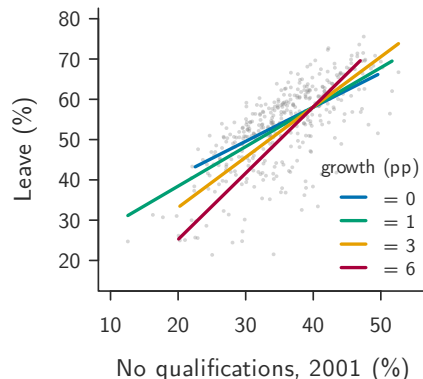
Leave, Qualifications, and Migration

- We can plot this data in a three-dimensional scatterplot.
- When we add in the regression surface, we see that it is quite complex...
- ...even when viewed from different angles.
- or when adding some shading to the surface.



Leave, Qualifications, and Migration

- As before, we can think of this as defining a regression line for no qualifications at every level of migrant growth.
- The function that describes how $E[Y|X_1, X_2]$ varies with X_1 given different levels of X_2 is given by:
$$\hat{y}_i = (\hat{\beta}_0 + \hat{\beta}_2 \cdot X_2) + (\hat{\beta}_1 + \hat{\beta}_3 \cdot X_2) \cdot X_1$$
- The conditional marginal effect of X_1 given different levels of X_2 is:
$$\frac{\partial \hat{Y}}{\partial X_1} = \hat{\beta}_1 + \hat{\beta}_3 \cdot X_2$$
- Here $\hat{Y} = 24.1 + 0.85 X_1 - 5.31 X_2 + 0.133 X_1 X_2$ (robust $t = 4.8$ on $X_1 X_2$). The slope of no qualifications is 0.85 at zero growth, 0.98 at 1 point, 1.25 at 3, and 1.65 at 6.



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Common Pitfalls

- ❶ Omission of lower order/component terms
- ❷ Misinterpretation of the coefficients on the lower order/component terms as “direct” effects
- ❸ Ignoring the covariance for computation of standard errors of conditional marginal effects
- ❹ Excessive extrapolation for the conditional effects
- ❺ False linearity assumptions

Read: Hainmueller, Mummolo, and Xu. 2019. “How much should we trust estimates from multiplicative interaction models? Simple tools to improve empirical practice.” *Political Analysis*.

Omission of lower order terms

Recall our interaction model

$$\text{Corruption} = \beta_0 + \beta_1 \text{Court} + \beta_2 \text{Dem} + \beta_3 (\text{Court} \cdot \text{Dem}) + u$$

- The fitted line for autocracies:

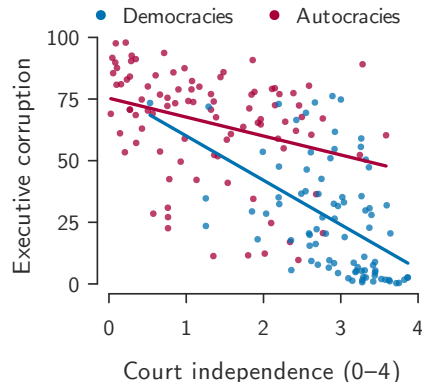
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X_{\text{court}}$$

$$\hat{y} = 75.3 - 7.7 X_{\text{court}}$$

- For democracies:

$$\hat{y} = (\hat{\beta}_0 + \hat{\beta}_2) + (\hat{\beta}_1 + \hat{\beta}_3) X_{\text{court}}$$

$$\hat{y} = 78.0 - 18.0 X_{\text{court}}$$



Suppose we instead omit the lower order term for democracy and fit

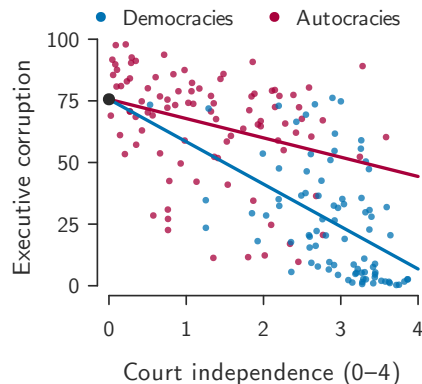
$$\text{Corruption} = \beta_0 + \beta_1 \text{Court} + \beta_3 (\text{Court} \cdot \text{Dem}) + u$$

Q. What do we assume?

- ① The predicted corruption level is the same in all democracies.
- ② The predicted corruption level is the same in all autocracies.
- ③ The slope of the regression line is the same in democracies and autocracies.
- ④ The intercept of the regression line is the same in democracies and autocracies.

Omission of lower order terms

- This model implies the following regression lines for democracies and autocracies.
- For autocracies:
$$Corr. = \beta_0 + \beta_1 Court$$
- For democracies:
$$Corr. = \beta_0 + (\beta_1 + \beta_3) Court$$
- Both lines must start from the same point at $Court = 0$. The fit hardly changes here (the full model's $\hat{\beta}_2 = 2.7$), but the interaction's standard error falls from 4.2 to 1.7 (t from -2.4 to -5.6). The restriction makes the evidence look much stronger than it is.
- In general, it is a bad idea to fit models that don't include all the lower order terms. These models impose strict and usually unrealistic assumptions.



Recall the interaction model with two continuous variables X_1 and X_2 :

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \cdot X_2) + u$$

How do we interpret β_1 and β_2 ? Marginal effects of X_1 and X_2 when the other variable is zero. In the Brexit model, $\hat{\beta}_1 = 0.85$ is the slope of no qualifications where migrant growth is zero, near many real areas (the lowest growth is 0.1). $\hat{\beta}_2 = -5.3$ is the slope of migrant growth where no one lacks qualifications. No area comes close (the lowest share is 12.6%).

Is it a good idea to omit lower order terms as in?

$$Y = \beta_0 + \beta_1 X_1 + \beta_3 (X_1 \cdot X_2) + u$$

$$Y = \beta_0 + \beta_1 X_1 + \beta_3 (X_1 \cdot X_2) + u$$

As before, it makes no sense to drop lower order terms from the interaction model:

- E.g., dropping no qualifications constrains its slope to be zero where migrant growth is zero.
- E.g., dropping migrant growth constrains its slope to be zero where no one lacks qualifications.
- In the Brexit data, dropping migrant growth changes the interaction from 0.13 ($t = 4.8$) to -0.005 ($t = -0.65$).
- **Never omit the lower order terms!!!**
 - Principle extends to higher order interactions (e.g. for three-way interactions you need to include all three main effects X_1 , X_2 , and X_3 , all two-way interaction terms ($X_1 \cdot X_2$, $X_1 \cdot X_3$, and $X_2 \cdot X_3$), and the three-way interaction term ($X_1 \cdot X_2 \cdot X_3$))
 - Lower order terms might only be omitted in special cases (e.g. when the lower order term is absorbed in the fixed effects)

Misinterpretation of the coefficients on the lower terms

Recall our interaction model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3(X_1 \cdot X_2) + u$$

The conditional marginal effect of X_1 is therefore conditional on the level of X_2 :

$$\frac{\partial Y}{\partial X_1} = \beta_1 + \beta_3 \cdot X_2$$

Therefore, β_1 is simply the marginal effect of X_1 at the point where $X_2 = 0$. Often, this is not a meaningful quantity of interest because X_2 might never take on the value of zero.

Never interpret the lower term coefficients β_1 or β_2 as “unconditional”, “direct”, or “independent” marginal effects of X_1 or X_2 !

The Adverse Effects of Sunshine?

	Question count (#)					Critical questions (%)		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Treated	0.016 (0.146)	0.247* (0.147)	0.249 (0.151)	0.271* (0.154)	0.942 (1.802)	3.188 (2.008)	3.013 (1.956)	2.799 (1.907)
Internet penetration		0.058 (0.035)	0.062* (0.033)	0.105 (0.070)		0.744 (0.578)	0.771 (0.593)	2.850*** (0.843)
Treated $\times$ Internet		-0.170*** (0.030)	-0.171*** (0.027)	-0.179*** (0.027)		-1.673*** (0.383)	-1.658*** (0.381)	-1.865*** (0.330)
Individual controls			✓	✓			✓	✓
Provincial controls				✓				✓
Observations	461	461	461	461	461	461	461	461

Table 5, panel A: change between Session 5 (Nov. 2009) and Session 6 (June 2010). OLS, standard errors clustered by province in parentheses. * $p < .1$, ** $p < .05$, *** $p < .01$.

E. Malesky, P. Schuler and A. Tran (2012). The Adverse Effects of Sunshine: A Field Experiment on Legislative Transparency in an Authoritarian Assembly. *American Political Science Review*, 106, pp 762-786.

“The component terms in the interaction are not robustly significant, but are uniformly positively signed. Nevertheless, we cannot say definitively that the transparency or Internet penetration has an independent effect on delegate participation.”

Ignoring the covariance for variance of conditional effects

Recall our interaction model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \cdot X_2) + u$$

The marginal effect of X_1 is therefore conditional on the level of X_2 :

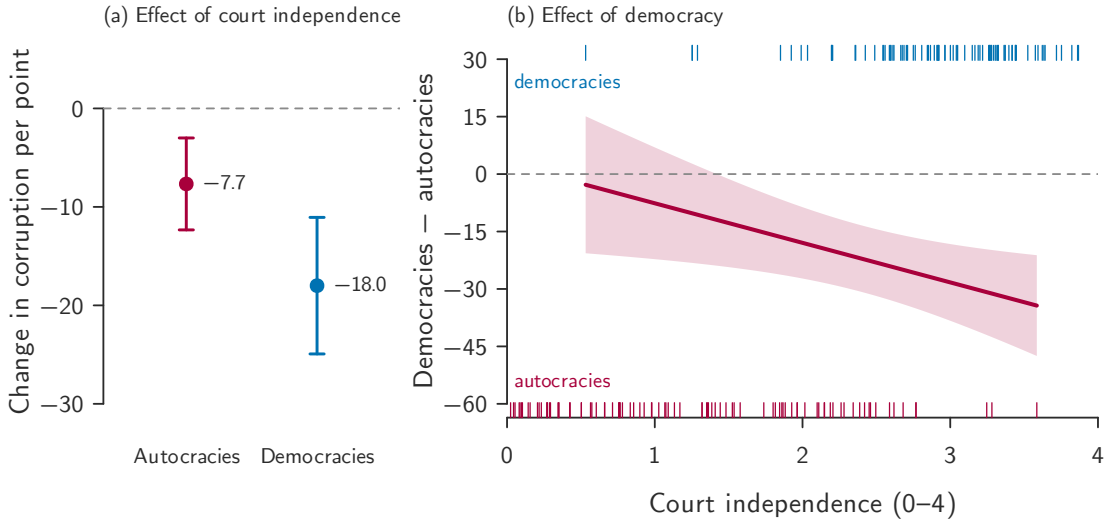
$$\frac{\partial \hat{Y}}{\partial X_1} = \hat{\beta}_1 + \hat{\beta}_3 \cdot X_2$$

The variance for the conditional effect of X_1 is therefore:

$$\text{Var} \left[\frac{\partial \hat{Y}}{\partial X_1} \right] = \text{Var}[\hat{\beta}_1 + \hat{\beta}_3 \cdot X_2] = \text{Var}[\hat{\beta}_1] + X_2^2 \text{Var}[\hat{\beta}_3] + 2X_2 \text{Cov}[\hat{\beta}_1, \hat{\beta}_3]$$

Never forget to include the covariance in computing the variance of the conditional effect

Corruption, Court Independence, and Democracy



Excessive extrapolation for the conditional effects

Recall our interaction model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3(X_1 \cdot X_2) + u$$

The marginal effect of X_1 is therefore conditional on the level of X_2 :

$$\frac{\partial \hat{Y}}{\partial X_1} = \hat{\beta}_1 + \hat{\beta}_3 \cdot X_2$$

Need to be careful to only compute this conditional effect for realistic values of X_2 (e.g. at the 25th, 50th, and 75th percentile of X_2). Computing the effect at levels of X_2 where there are no or only few observations requires excessive extrapolation and results in high model dependency.

Never compute the conditional effect of the variable of interest at unrealistic or atypical values of the conditioning variable. Always show the distribution of the conditioning variable in the plot (e.g. using a rugplot).

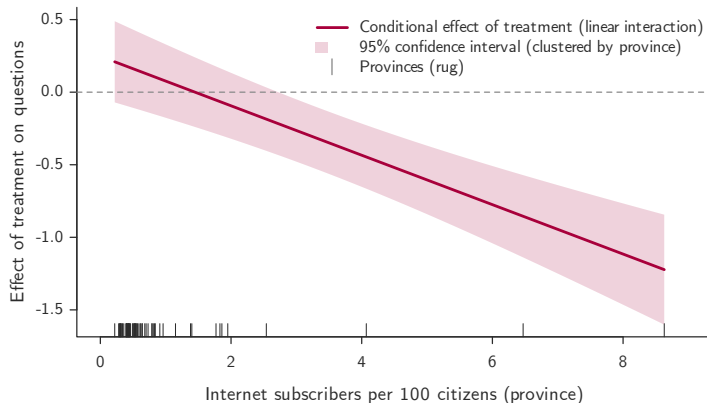
The Adverse Effects of Sunshine?

E. Malesky, P. Schuler and A. Tran (2012). The Adverse Effects of Sunshine: A Field Experiment on Legislative Transparency in an Authoritarian Assembly. *American Political Science Review*, 106, pp 762-786.

An online newspaper, VietnamNet, reported the legislative activity of a randomly chosen group of delegates to Vietnam's National Assembly. Their Figure 1 plots the estimated effect of this treatment on questions asked and on criticism against Internet subscribers per 100 citizens in the delegate's province: near zero where few people are online, increasingly negative where more are.

“We find no evidence of a direct effect of the transparency treatment on delegate performance; however, further analysis reveals that delegates subjected to high treatment intensity demonstrate robust evidence of curtailed participation and damaged reelection prospects. These results make us cautious about the export of transparency without electoral sanctioning.”

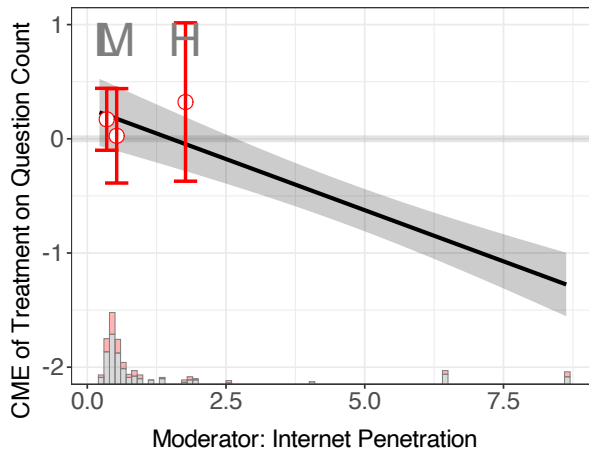
The Adverse Effects of Sunshine?



J. Anderson (2013). Sunshine Works. Comment on “The Adverse Effects of Sunshine: A Field Experiment on Legislative Transparency in an Authoritarian Assembly.” *World Bank Policy Research Working Paper* 6602.

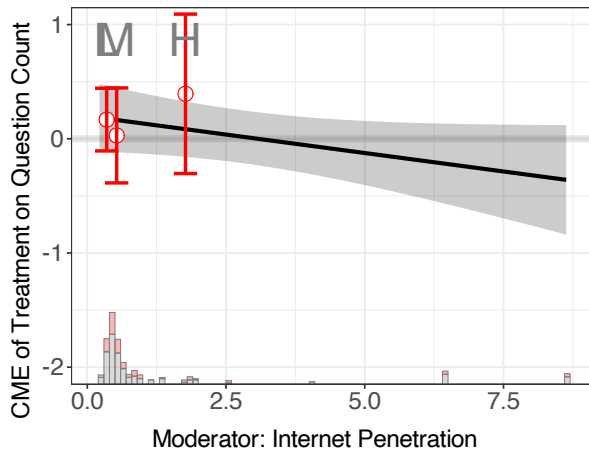
On the change in *critical* questions: “there are only three provinces (Hanoi, HCMC, and Da Nang) out of 64 (i.e., fewer than 5 percent of provinces) for which the predicted effect of treatment is statistically significantly different from zero at any conventional level.”

The Adverse Effects of Sunshine?



Source: Replicates Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Figure 6(b).

The Adverse Effects of Sunshine?



Source: Replicates Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Figure 6(c), after dropping four high-leverage delegates.

Excessive extrapolation for the conditional effects

Recall our interaction model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \cdot X_2) + u$$

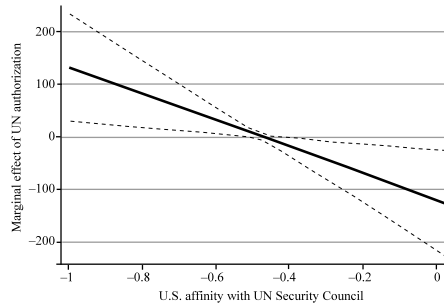
The marginal effect of X_1 is therefore conditional on the level of X_2 :

$$\frac{\partial \hat{Y}}{\partial X_1} = \hat{\beta}_1 + \hat{\beta}_3 \cdot X_2$$

Another issue you need to check when computing the conditional effect of X_1 is whether at the realistic values of X_2 there is actually sufficient variation in X_1 . Computing the effect at levels of X_2 where there is no or little variation in X_1 requires excessive extrapolation and results in high model dependency.

Never compute the conditional effect of the variable of interest at values of the conditioning variable where there is no or almost no variation on the variable of interest. Always check the distribution of the variable of interest across the values of the conditioning variable for which you plot the effect.

Audience Beliefs and IO Legitimacy?



Note: Dashed lines give 95 percent confidence interval.

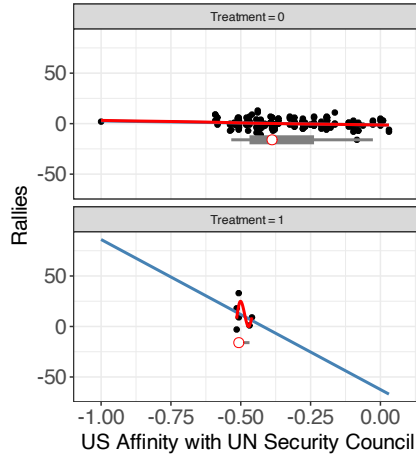
FIGURE 2. *Marginal effect of UN authorization by affinity with the Security Council*

Chapman, T. 2009. Audience Beliefs and International Organization Legitimacy. *International Organization*. 63(4):733-764.

Looks at the effect of UN Security Council authorization for use of force on changes in presidential approval ("rallies") after military onset and how this effect depends on the similarity of preferences between the U.S. and the UN Security Council.

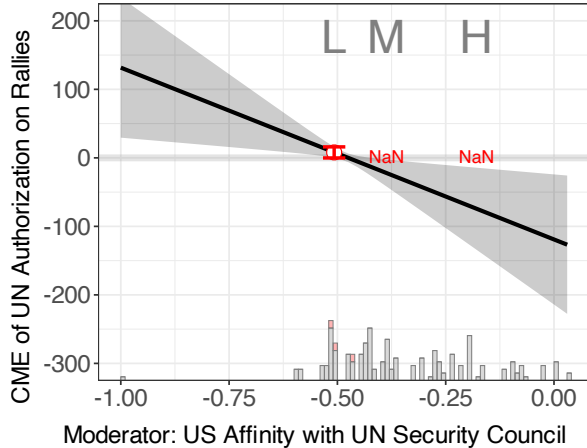
"Clearly, the effect of authorization on rallies decreases as similarity increases."

Audience Beliefs and IO Legitimacy?



Source: Replicates Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Figure 5(a).

Audience Beliefs and IO Legitimacy?



Source: Replicates Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Figure 5(b).

False linearity assumption

Recall our interaction model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3(X_1 \cdot X_2) + u$$

The marginal effect of X_1 is therefore conditional on the level of X_2 :

$$\frac{\partial Y}{\partial X_1} = \beta_1 + \beta_3 \cdot X_2$$

The functional form here assumes that the interaction effect is linear, i.e. the marginal effect of X_1 changes linearly with levels of X_2 . But this linearity assumption is often flawed. For example, the effect of X_1 might be large at low levels of X_2 , small at medium levels of X_2 , and large at high levels of X_2 .

Always check the linearity assumption by running a more flexible model that allows the conditional effect to vary more freely across levels of the conditioning variable.

Relaxing the Linearity Assumption

Discretize the conditioning variable X_2 into three equal sized bins using three dummy variables G_j with $j = 1, 2, 3$:

$$G_{i1} = \begin{cases} 1 & X_{2i} < \delta_{1/3} \\ 0 & \text{otherwise} \end{cases}, \quad G_{i2} = \begin{cases} 1 & X_{2i} \in [\delta_{1/3}, \delta_{2/3}) \\ 0 & \text{otherwise} \end{cases}, \quad G_{i3} = \begin{cases} 1 & X_{2i} \geq \delta_{2/3} \\ 0 & \text{otherwise} \end{cases},$$

in which $\delta_{1/3}$ and $\delta_{2/3}$ are respectively the first and second tertiles of X_2 .

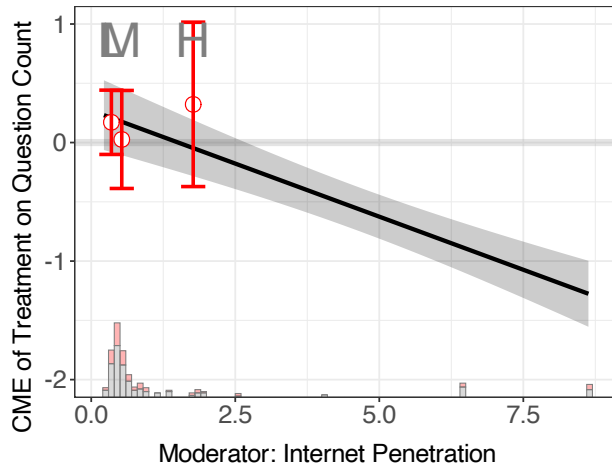
Let $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ be the median values of X_2 in each of the three bins, respectively.

We then estimate the following model:

$$Y = \sum_{j=1}^3 \{ \mu_j + \eta_j(X_2 - \bar{x}_j) + \alpha_j X_1 + \lambda_j(X_2 - \bar{x}_j)X_1 \} G_j + u$$

where α_j is the conditional effect of X_1 at the bin median $X_2 = \bar{x}_j$.

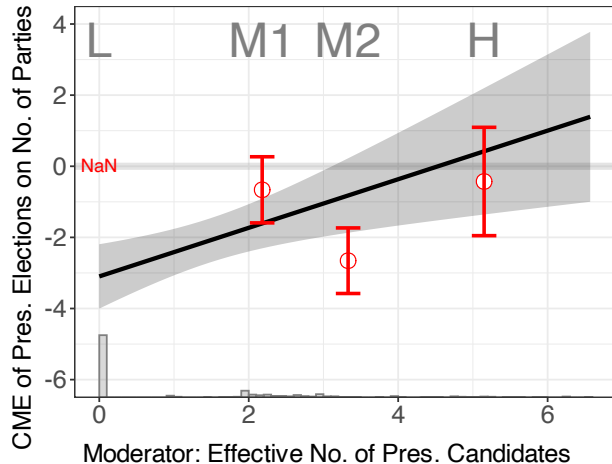
The Adverse Effects of Sunshine?



- The bin estimates at low, medium and high Internet penetration are 0.17, 0.03 and 0.32. None differs from zero at the 5% level.
- The bin estimates do not fall as the line does. The line's large negative effects lie where the histogram shows almost no delegates.

Source: Malesky, Schuler & Tran (2012) data, with their covariates; SEs clustered by province. Replicates Hainmueller, Mummolo & Xu (2019), Figure 6(b).

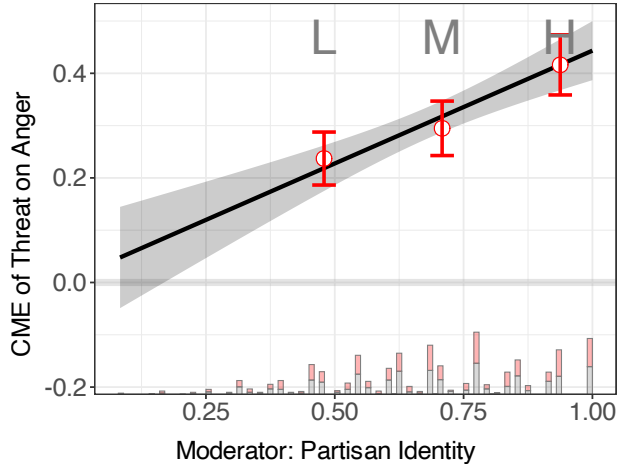
Linear Interaction? Clark and Golder (2006)



- Bin L has no estimate. With no presidential election ($X = 0$), the treatment is always 0.
- The effect is -2.66 at M2, far below the line, and its interval excludes zero. At M1 and H it is near zero. The effect does not change linearly with X .

Source: Clark & Golder (2006) data; SEs clustered by country. Replicates Hainmueller, Mummolo & Xu (2019), Figure 7(a).

Linear Interaction! Huddy et al (2015)



- The bin estimates rise with partisan identity, from 0.24 to 0.29 to 0.42. Each differs from zero.
- Each is close to the line. Here the linear interaction model holds up.

Source: Huddy, Mason & Aarøe (2015) data; robust SEs. Replicates Hainmueller, Mummolo & Xu (2019), Figure 4(b).

A More Flexible Estimator: The Kernel Estimator

Binning estimates the effect at three points. The **kernel estimator** (Hainmueller, Mummolo & Xu 2019, Section 4.2) estimates it across the whole range of X in the model

$$Y = f(X) + g(X)D + \gamma(X)Z + \varepsilon,$$

where f , g and γ are smooth functions of X and $g(X)$ is the conditional marginal effect of D . The linear interaction model is the special case $f(X) = \mu + \eta X$, $g(X) = \alpha + \beta X$, $\gamma(X) = \gamma$.

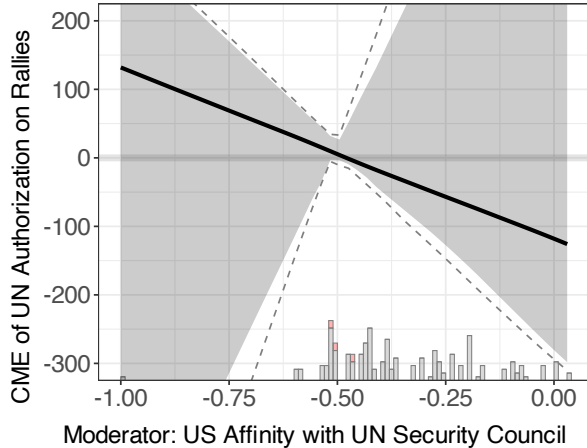
For each evaluation point x_0 , fit the interaction model **locally**, by weighted least squares:

$$\min_{\mu, \alpha, \eta, \beta, \gamma} \sum_{i=1}^N [Y_i - \mu - \alpha D_i - \eta(X_i - x_0) - \beta D_i(X_i - x_0) - \gamma Z_i]^2 K\left(\frac{X_i - x_0}{h}\right),$$

and set $\hat{g}(x_0) = \hat{\alpha}(x_0)$. Repeating this over a grid of x_0 traces out $\hat{g}(\cdot)$.

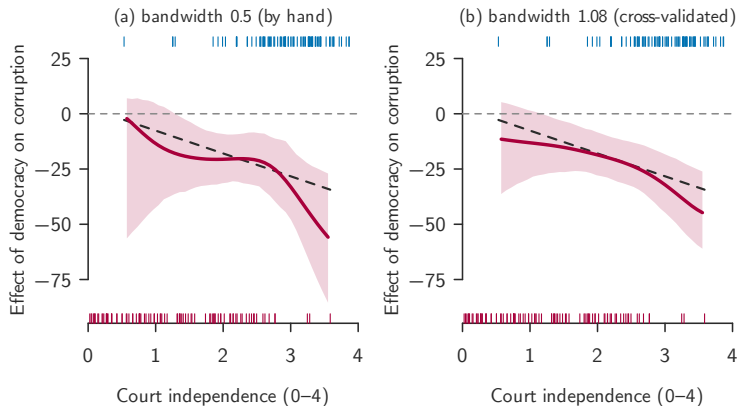
- K is a Gaussian kernel: units with X_i near x_0 get the most weight.
- The bandwidth h is chosen by cross-validation; `interflex` makes it wider where the data on X are sparse.
- Confidence intervals come from the bootstrap (a block bootstrap with clusters).
- Newer versions of `interflex` also offer double/debiased machine learning (DML) estimators (Liu, Liu & Xu 2025), covered in 450B.

Audience Beliefs and IO Legitimacy?



Source: Replicates Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Figure 5(c), bandwidth by cross-validation (0.34; their code: 0.30).

Kernel Estimator: Corruption, Courts, and Democracy



- With $h = 0.5$ the effect stays near -20 from court independence 1.4 to 2.5, then drops past 2.8, where only three autocracies sit. The interval is wide at both ends.
- Cross-validation picks $h = 1.08$, whose curve stays close to the line. Every h from about 0.4 up fits about as well (loss 8.54 at $h = 0.49$, 8.46 at 1.08, 8.53 at 3.84; SE 0.78). The kernel estimates do not contradict the linear model here.

Effect of democracy (D) on executive corruption across court independence (X), with pointwise 95% bootstrap intervals (500 resamples), drawn where both groups have data. Dashed: the linear interaction model. Ticks: autocracies (bottom), democracies (top).

Source: V-Dem v16, 2024.

How Much Should We Trust Linear Interaction Models?

Journal	N	Not rejecting same effect at low vs. high	Severe extrapolation	Rejecting linear model	Mean score
AJPS	9	0.78	0.78	0.22	1.8
APSR	17	0.71	0.41	0.65	1.8
CPS	10	0.67	0.3	0.5	1.3
IO	7	0.83	0.71	0.67	2.3
JOP	3	0.33	0.33	0.67	1.3
All journals	46	0.71	0.50	0.52	1.7

Replications of 46 published interaction effects (the unit is the interaction, not the article). Each of the three middle columns is the share of effects with that problem; the mean score is the average number of these problems per effect, over the 41 effects for which all three checks were possible. The low vs. high and linearity checks were possible for 44 and 42 effects.

Source: Hainmueller, Mummolo & Xu (2019), *Political Analysis* 27(2), Table 1.

Interaction Terms

Polynomials

- Polynomial terms on the RHS in a regression equation can be seen as a special case of the continuous interactions of the previous section.
- For example, when $X_1 = X_2$ in the previous interaction model, we get a quadratic on the RHS:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + u$$

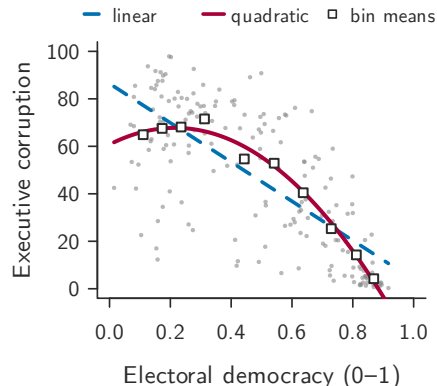
$$Y = \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u$$

$$Y = \beta_0 + \gamma_1 X_1 + \gamma_2 X_1^2 + u, \quad \gamma_1 = \beta_1 + \beta_2, \quad \gamma_2 = \beta_3$$

- This is called a **second order polynomial** in X_1
- A **third order polynomial** is given by: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + u$

Polynomial Example: Corruption and Democracy

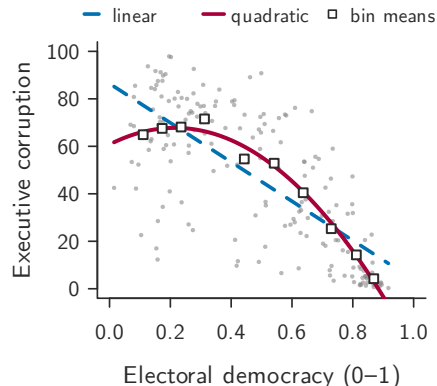
- Let's look at the 179 countries of the running example and examine the relationship between **Y: executive corruption** and **X: electoral democracy** (V-Dem's index, 0–1)
- We see that a simple linear specification does not fit the data very well:
$$Y = \beta_0 + \beta_1 X_1 + u$$
- A second order polynomial in democracy fits the data better (R^2 0.60 vs. 0.53):
$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$$



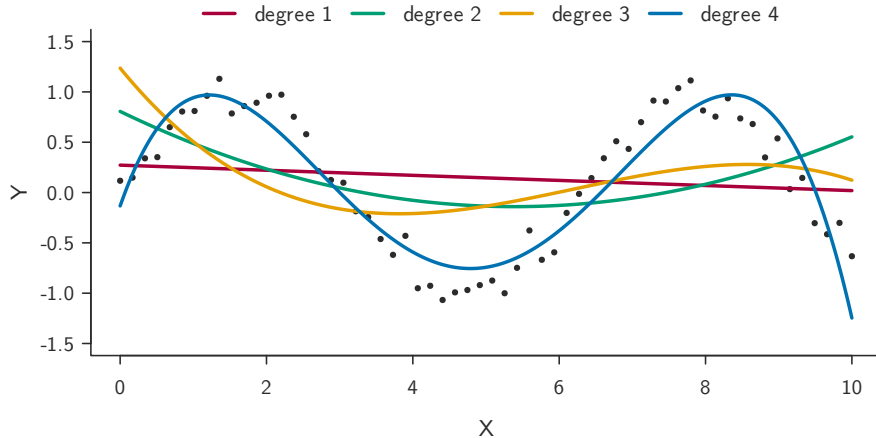
Source: V-Dem v16, 2024 (executive corruption index $\times$ 100; electoral democracy index). Squares: mean corruption in each tenth of the countries, sorted by democracy.

Polynomial Example: Corruption and Democracy

- $\hat{Y} = 60.8 + 65.1X_1 - 151.5X_1^2$
- Is β_1 the marginal effect of democracy on corruption?
- No! The slope depends on the level of democracy:
 $\frac{\partial \hat{Y}}{\partial X_1} = \hat{\beta}_1 + 2\hat{\beta}_2 X_1$
Here it is positive at low democracy and turns more and more negative as democracy rises.
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $\frac{-\beta_1}{2\beta_2}$. Here the turning point is at $X_1 \approx 0.215$. But the bin means show no real rise. They stay between 65 and 72 up to about 0.37, then fall to 4. So the data are roughly flat, then falling.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

Dealing with Polynomials in R

- In R, you must enclose powers in a “I()” construct inside a formula (here d holds the 179 countries: execorr100 is executive corruption, polyarchy electoral democracy):

```

R Code
> summary(lm(execorr100 ~ polyarchy + I(polyarchy^2), data = d))

Call:
lm(formula = execorr100 ~ polyarchy + I(polyarchy^2), data = d)

Residuals:
    Min       1Q   Median       3Q      Max
-54.973  -9.414   2.149  13.340  37.712

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    60.765     5.425   11.200 < 2e-16 ***
polyarchy       65.118    26.670    2.442  0.0156 *
I(polyarchy^2) -151.461    26.780   -5.656 6.17e-08 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 18.92 on 176 degrees of freedom
Multiple R-squared:  0.6014,    Adjusted R-squared:  0.5969
F-statistic: 132.8 on 2 and 176 DF,  p-value: < 2.2e-16
```