

Lecture 2-2: Interval Estimation

Goal: Obtain confidence intervals using normal or t approximations.

- Interval estimation
 - Point vs. interval estimates
 - Interpreting confidence intervals
 - Coverage rate
- Normal mean with known variance
 - Standardization with known σ
 - z -based confidence intervals
 - Choosing confidence levels
- Normal mean with unknown variance
 - Estimating σ^2 via sample variance
 - Student's t distribution
 - t -based confidence intervals
- Mean with unknown distribution
 - Central limit theorem justification
 - Asymptotic normal intervals
 - Prefer t for small samples
- Practical guidance
 - When to use z vs. t
 - Conservative choices by sample size

Worksheet

1. (Two variance estimators) Define $S_{0n}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $S_{1n}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$. Show:

i) $E[S_{0n}^2] = \frac{n-1}{n} \sigma^2$ and $E[S_{1n}^2] = \sigma^2$.

- ii) Both estimators are consistent. (Hint: write $S_{0n}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 - (\bar{X}_n - \mu)^2$ and apply the WLLN to each term.)

2. (t statistic) With unknown σ^2 and $X_i \sim N(\mu, \sigma^2)$, prove $T_n = \frac{\bar{X}_n - \mu}{S_{1n}/\sqrt{n}} \sim t_{n-1}$. (Hint: If $Z \sim N(0, 1)$ and $U \sim \chi_\nu^2$ are independent, then $T = \frac{Z}{\sqrt{U/\nu}} \sim t_\nu$, which defines the t -distribution. Under normality, $\sqrt{n}(\bar{X}_n - \mu)/\sigma \sim N(0, 1)$ and $(n-1)S_{1n}^2/\sigma^2 \sim \chi_{n-1}^2$, and the two are independent.)

3. (CI for linear function of μ) Suppose we have an i.i.d. random sample of size n , $X_1, \dots, X_n$, from $X \stackrel{iid}{\sim} N(\mu, \sigma^2)$. For constants a, b with $b \neq 0$, derive a $100(1 - \alpha)\%$ CI for $a + b\mu$ under (i) known σ^2 , (ii) unknown σ^2 .