

Lecture 6-1: OLS with Matrix Algebra

Goal: Derive OLS estimator in matrix form and study its properties.

- Preparation
 - Matrix representation of regression
 - Solving systems of equations
- Linear model in matrix form
 - Vector derivatives
 - Derivation of OLS estimator
 - OLS assumptions in matrix notation
 - Properties of OLS estimator
- Frisch-Waugh-Lovell Theorem
 - Projection and annihilator matrices
 - Proving FWL

Worksheet

1. With $n = 3$, regress y on an intercept and x where

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

- (a) Compute $\hat{\boldsymbol{\beta}}$, fitted values $\hat{\mathbf{y}}$, residuals $\hat{\mathbf{u}}$.
- (b) Verify the normal equations $\mathbf{X}'\hat{\mathbf{u}} = \mathbf{0}$.
- (c) Compute $SSR = \hat{\mathbf{u}}'\hat{\mathbf{u}}$ and $\hat{\sigma}^2 = SSR/(n - (k + 1))$.

2. Under homoskedasticity and independent sampling, $V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I}_n$.

- (a) Prove that $V[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$.
- (b) For any fixed $(k+1) \times 1$ vector $\mathbf{a}$, derive $V[\mathbf{a}'\hat{\boldsymbol{\beta}}|\mathbf{X}]$ and the plug-in standard error.
- (c) Return to the data in Question 1 and report the estimated variance-covariance matrix $\hat{V}[\hat{\boldsymbol{\beta}}] = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$.

3. Suppose the true model is $y_i = \mathbf{x}_i\boldsymbol{\beta} + u_i$ with $E[u_i|\mathbf{x}_i] = 0$, and the sample $\{(y_i, \mathbf{x}_i)\}_{i=1}^n$ is i.i.d. with $E[\mathbf{x}_i'\mathbf{x}_i]$ finite and nonsingular.
- (a) Show that $\hat{\boldsymbol{\beta}}$ is a consistent estimator of $\boldsymbol{\beta}$.
 - (b) Assume further that $E[u_i^2|\mathbf{x}_i] = \sigma^2 < \infty$, derive the asymptotic distribution of $\hat{\boldsymbol{\beta}}$.

4. Assume the classical linear regression model satisfying Assumptions I–V:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u},$$

where $\mathbf{X}$ is an $n \times k$ matrix of full column rank, and $\mathbf{y}$ is an $n \times 1$ vector of responses.

- (a) What does $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ represent in the context of OLS estimation?
- (b) What is the interpretation of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$?
- (c) What does $\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ represent?
- (d) What does $[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]$ compute?
- (e) What does $[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]/(n - k)$ estimate?
- (f) What is the interpretation of

$$(\mathbf{X}'\mathbf{X})^{-1} \frac{[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]'[\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}]}{(n - k)} ?$$

5. Following the same setup in the previous question, define:

$$\mathbf{Q} = \mathbf{X}'\mathbf{X}, \quad \mathbf{A} = \mathbf{Q}^{-1}\mathbf{X}', \quad \mathbf{P} = \mathbf{X}\mathbf{A}, \quad \mathbf{M} = \mathbf{I} - \mathbf{P}.$$

Here, $\mathbf{P}$ is the projection (hat) matrix and $\mathbf{M}$ is the residual maker matrix.

- (a) Express the residuals from regressing $\mathbf{y}$ on $\mathbf{X}$ as a function of $\mathbf{M}$.
- (b) Show that $\mathbf{P}$ is symmetric and idempotent.
- (c) Show that $\mathbf{M}$ is idempotent (verify $\mathbf{M}'\mathbf{M} = \mathbf{M}$ and $\mathbf{M}^2 = \mathbf{M}$).
- (d) Show that $\mathbf{A}\mathbf{P} = \mathbf{A}$, $\mathbf{A}\mathbf{M} = \mathbf{0}$, $\mathbf{M}\mathbf{P} = \mathbf{0}$, and $\mathbf{P}\mathbf{M} = \mathbf{0}$.
- (e) Show that $\mathbf{P}\mathbf{X} = \mathbf{X}$, $\mathbf{M}\mathbf{X} = \mathbf{0}$.
- (f) Show that $\mathbf{P}\hat{\mathbf{y}} = \hat{\mathbf{y}}$, $\mathbf{P}\hat{\mathbf{u}} = \mathbf{0}$.
- (g) Show that $\mathbf{M}\hat{\mathbf{y}} = \mathbf{0}$, $\mathbf{M}\hat{\mathbf{u}} = \hat{\mathbf{u}}$.
- (h) Show that $\mathbf{X}'\hat{\mathbf{y}} = \mathbf{X}'\mathbf{y}$.
- (i) Show that $\hat{\mathbf{u}}'\hat{\mathbf{u}} = \mathbf{y}'\mathbf{M}\mathbf{y} = \mathbf{y}'\mathbf{y} - \hat{\mathbf{y}}'\hat{\mathbf{y}}$.

Hint: Many of the results in this problem build on earlier parts. Use previously shown equalities (e.g., $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}' = \mathbf{P}$, $\mathbf{M} = \mathbf{I} - \mathbf{P}$) to simplify the algebra.